International Journal of Advanced Research in Engineering and Technology (IJARET)
Volume 12, Issue 5, May 2021, pp.13-24 Article ID: IJARET_12_05_002
Available online at https://iaeme.com/Home/issue/IJARET?Volume=12&Issue=5
ISSN Print: 0976-6480 and ISSN Online: 0976-6499
DOI: 10.34218/IJARET.12.5.2021.002

PERFECT FOLDING OF A SURFACE TO A POLYGON

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ABSTRACT

In this paper, we introduced the definition of the perfect folding of surfaces, also we developed the theory of cellular and perfect foldings of a compact surface onto polygons. Our main interest is to know whether and how many cellular and perfect foldings of a given surface onto a given polygon do exist.

2010 Mathematics subject classification: 51H05, 51H20, 51H99

Key words: Surface, topological polygon, graph, regular folding.

Cite this Article: E. M. El-Kholy and H. Ahmed, Perfect Folding of a Surface to a Polygon, *International Journal of Advanced Research in Engineering and Technology (IJARET)*, 12(5), 2021, pp. 13-24.
https://iaeme.com/Home/issue/IJARET?Volume=12&Issue=5

1. INTRODUCTION

Throughout this paper, we use the term surface to mean a compact connected topological 2-manifold without boundary. A *cell decomposition* C of a surface M is a partition of M into disjoint open cells such that for each cell σ of C, its closure $\bar{\sigma}$ is a closed cell, that is if σ is an n-cell, n=1,2, then $\bar{\sigma}$ is homeomorphic to the unit closed sphere $\bar{D}^n$. A closed 2-cell is called a face of C, a closed 1-cell is an *edge* and a 0-cell a *vertex*. Let M be a surface, a continuous map $f: M \to P_n$ of M onto an n-gon P_n is called a cellular folding if there is a finite *cell decomposition* C_f of M such that

(1) f is a cellular map of C_f onto $C(P_n)$.

(2) For each closed cell $\bar{\sigma}$, the restriction map $f\big|\bar{\sigma}$ is a homeomorphism of $\bar{\sigma}$ onto a closed cell $\bar{\tau}$ of $C(P_n)$, [1].

To avoid trivial cases, we require that each 0-cell is an end-point of more than two 1- cells. Thus for a cellular folding $f: M \to P_n$ the *edges* and *vertices* of C_f form a finite graph Γ_f

embedded in M without loops (but possibly with multiple *edges*) and f "folds" M along the *edges* of Γ_f . For each *vertex* v of Γ_f the number of 1-cells of Γ_f having v as an end point is called the *valency* of v. It should be noted that for any cellular folding f, every vertex of Γ_f has even *valency*,[2]. The cellular folding (or Γ_f) is said to be regular if all the *vertices* have the same *valency*. A *regular* folding onto P_n with valency k is called a *regular* folding of type (k, n).

We denote by M_g and N_p an orientable surface of genus g and a non-orientable surface of genus p respectively. The start point of the study of *regular* foldings of a surface onto polygons was given by the paper of H. R. Farran, E. EL-Ekholy and S. A. Robertson, [2].

Proposition (1-1), [2]

For each cellular folding $f: M \to P_n$ with α *vertices*, β *edges* and γ *faces* we have

(1) $n\gamma = 2\beta$.

(2) $\alpha - \beta + \gamma = e(M)$.

(3) $n\gamma \geq 4\alpha$ (each vertex has *valency* ≥ 4).

(4) The Euler characterisitic $e(M) \leq \alpha((4 \div n) - 1)$.

If f is a *regular* folding of type (k, n) , we have in addition

(5) $k\alpha = n\gamma = 2\beta$.

(6) If $M = M_g$ is an orientable surface with genus g , then

$g = 1 + \frac{(k-2)(n-2)-4}{4n} \alpha$.

(7) If $M = N_p$ is a non-orientable surface with genus p , then

$p = 2 + \frac{(k-2)(n-2)-4}{2n} \alpha$.

From these relations, they obtained some pairs of surfaces and polygons between which there are no regular foldings. They also classified all the possible quintuplets $(k, n, \alpha, \beta, \gamma)$ of the above five numbers associated to regular foldings of a double torus onto polygons. In this paper, we discover a new additional relation that must be satisfied by the quintuplet $(k, n, \alpha, \beta, \gamma)$. Using this we obtain non existence theorems for regular and perfect foldings between a wide range of pairs of surfaces and polygons.

2. ADDITIONAL CONDITIONS FOR REGULAR FOLDING

One might expect that for every quintuplet $(k, n, \alpha, \beta, \gamma)$ which satisfies the conditions in Proposition (1-1) there exist a *regular* folding with the quintuplet $(k, n, \alpha, \beta, \gamma)$. However this is not the case in general.

Example(2-1)

The quintuplet $(k, n, \alpha, \beta, \gamma)$=(8,4,10,40,20) satisfies all the conditions in Proposition (1.1). However there is no *regular* folding $f: M_6 \to P_4$ with $(k, n, \alpha, \beta, \gamma)$= (8,4,10,40,20), [4]. Our first result is the discovery of an additional relation between elements of the quintuplet to have a regular folding which states that n divideds α .

Theorem(2-2)

Let $f: M \to P_n$ be a *regular* folding with quintuplet $(k, n, \alpha, \beta, \gamma)$ then the following properties hold.

(i)Let A_1 , A_2 , … , A_n be the vertices of P_n . For any two distinct vertices $v, w \epsilon f^{-1}(A_i)$, there exist two distinct open 2-cells σ and τ such that $v \epsilon \bar{\sigma}$, $w \epsilon \bar{\tau}$.

(ii) For all i = 1, … , n , $\#f^{-1}(A_i) = \gamma/k$.

(iii) ' k divides γ ' .

(iv)' n divides α '.

Proof :

(i)Let A_i be any vertex of P_n and let $v, w \epsilon f^{-1}(A_i)$ with $v \neq w$. Then there exist two 2-cells σ and τ such that $v \epsilon \bar{\sigma}$, $w \epsilon \bar{\tau}$. Now suppose $\sigma = \tau$, then σ has v and w as vertices at the same time. However $f\big|\bar{\sigma} : \bar{\sigma} \to P_n$ must be a homeomorphism and hence $f(v) \neq f(w)$, which contradicts the assumption that $f(v) = f(w) = A_i$. So $\sigma \neq \tau$ and $\sigma \cap \tau = \emptyset$.

(ii) Let σ_i, i=1,2,…, γ be the 2-cells of f, so each $\bar{\sigma_i}$ is homeomorphic to P_n. We cut M into $\overline{\sigma_1}$, $\overline{\sigma_2}$,…, $\overline{\sigma_\gamma}$. If we count vertices independently, then we have γ vertices which go to A_i, one on each 2-cell. However each vertex has valency k , so each vertex is counted k times. Therefore the number of vertices in $f^{-1}(A_i)$ is γ/k .

(iii) From (ii) , γ/k is an integer. Therefore k divideds γ .

(iv)From (iii) $\gamma/k = l$, for some integer l. On the other hand, from Proposition (1.1), we have $k\alpha = n\gamma$ which implies that $\alpha = n(\gamma/k)$, which implies that n divideds α because γ/k is an integer. □

Now the quintuplet $(k, n, \alpha, \beta, \gamma)$ in Example (2-1) does not satisfy condition (iv) in Theorem (2-2). Hence there is no regular folding $f: M_6 \to P_4$ with $(k, n, \alpha, \beta, \gamma)$= (8,4,10,40,20).

The next result follows directly from Theorem (2-2).

Corollary (2-3)

For each regular n-folding $f: M \to P_n$ of type (k, n) with $n \geq 3$, we have:

If M is an orientable surface of genus g , then

(i)The number α/n is a positive integer, $\frac{\alpha}{n} = \frac{4(g-1)}{(k-2)(n-2)-4}$

(ii) $4(g-1) \geq (k-2)(n-2) - 4$

(iii) $k \leq \frac{4g}{n-2} + 2$

If $M$is a non-orientable surface with genus p , then

(iv) The number α/n is a positive integer, $\frac{\alpha}{n} = \frac{2(p-2)}{(k-2)(n-2)-4}$

(v) $2(p-2) \geq (k-2)(n-2) - 4$

(vi)) $k \leq \frac{2p}{n-2} + 2$

3. PERFECT FOLDINGS OF A SURFACE TO A POLYGON

Let $f: M \to P_n$ be a regular folding and and C_f be the cell decomposition of M**.** Let $H(f)$ be the set of homeomorphisms $h: M \to M$ which are also cellular maps of C_f at the same time. $H(f)$ becomes a group with respect to the composition of homeomorphisms. An isometry of M is called an isometry of f, or an automorphism of f if it preserves the cellular decomposition of f. Then the automorphisms of f form a group with respect to the composition of isometries, which we call the group of automorphisms of f or the group of symmetries of f and we denote it by $G(f)$.

In order to investigate the action on C_f of the group of homeomorphism$H(f)$, it is enough to investigate the action of the group of isometries $G(f)$ on C_f , [4]. Since the surface M under consideration is compact, the number of cells of C_f is finite. If the action of two isometries h_1 and h_2 on C_f are the same, then $h_1 = h_2$. Thus the number of isometries which preserve C_f is less than or equal to the number of bijections of C_f which is ($\alpha!\ \beta!\ \gamma!$) , where α, β and γ are the number of vertices, the edges and the 2-cells of C_f respectively. Thus the group of isometries of M preserving C_f is a finite group. On the other hand the order of the group of homeomorphisms which preserve C_f is unnecessarily huge, actually it has the same number as the set of real numbers which we do not think simple and beautiful.

Definition (3-1)

A regular folding $f: M \to P_n$ is called perfect if the automorphism group $G(f)$ acts transitively on the 2- cells of the cell decomposition.

Example (3-2)

Consider $M = S^2$ with a cellular subdivision consisting of six 0-cells, twelve 1-cells and four 2-cells. Let $f: S^2 \to P_3$ be a cellular folding defined by $f\ \{v_1, \ldots, v_6\} = \{v_1, v_2, v_3, v_2, v_3, v_1\}$, $f\{e_1, \ldots, e_{12}\} = \{e_1, e_1, e_1, e_1, e_5, e_5, e_5, e_5, e_{10}, e_{10}, e_{10}, e_{10}\}$. In this case the graph Γ_f is a regular graph isomorphic to the edge graph of the octahedron, and we call f the octahedra folding " w " of S^2. The octahedron is one of the five platonic regular polyhedra. The group $G(w)$ of the octahedral regular folding is isomorphic to $\mathbb{Z}_2 \times G_4$, where G_4 is the group of permutations of four letters, [3]. This group acts transitively on the set of 2-cells. Thus f is a perfect folding, see Fig.(1).

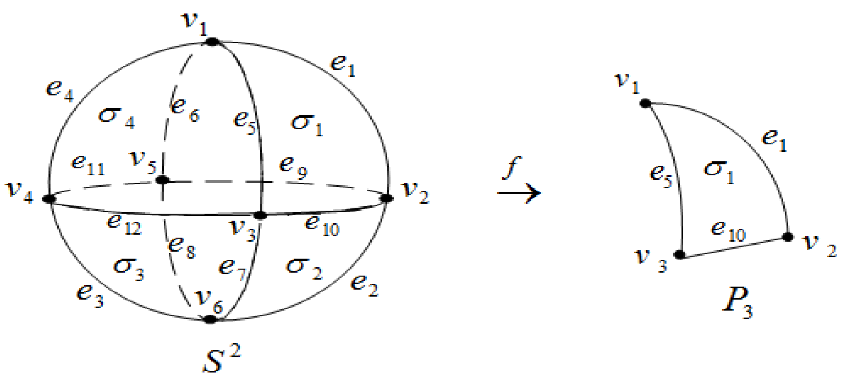

Figure 1

We now give an example of a regular folding whose group of automorphisms does not acts transitively on the 2-cells of the associated cell decomposition. First we define the sense in which two cellular foldings are to be regarded as equivalent to on another.

Definition (3-3)

Let $f: M \to P_r$ and $g: M \to P_s$ be cellular foldings. Then we say that f is topologically equivalent to g and we write $f \approx g$ iff there are homeomorphisms $h_1: M \to N$ and $h_2: P_r \to P_s$

such that $g \circ h_1 = h_2 \circ f$. It follows at once that $f \approx g$ iff there is a homeomorphism $h_1: M \to N$ such that $h_1(\Gamma_f) = \Gamma_g$, where Γ_f and Γ_g are the graphs associated to regular foldings f and g respectively. Hence $h_1 | \Gamma_f$ is a graph isomorphism onto Γ_g . Also $f \approx g$ implies $r = s$.

Example (3-4)

There are two topological types of regular foldings of N_3 to P_3 with $(k, n, \alpha, \beta, \gamma)$ = (8,3,3,12,8).

First, we explain our method to construct regular foldings. Suppose that we have a regular folding $f: N_3 \to P_3$ with $(k, n, \alpha, \beta, \gamma)$ = (8,3,3,12,8). Let C_f denote the cell decomposition of N_3 associated to f. Let u, v and w be the vertices of P_3 and let u', v', w' be the corresponding vertices of C_f ,i.e., $f(u') = u$, $f(v') = v$, $f(w') = w$. Since $(k, n, \alpha, \beta, \gamma)$ = (8,3,3,12,8), all the eight 2-cells, σ_i , i=1,…, 8 of C_f has u'as a vertex. Let a_i be the edge in σ_i facing to the vertex u'. Cutting the surface N_3 along the edges a_1 , a_2 , … , a_8 , we obtain an extension figure round the vertex u' such as Fig.(2).

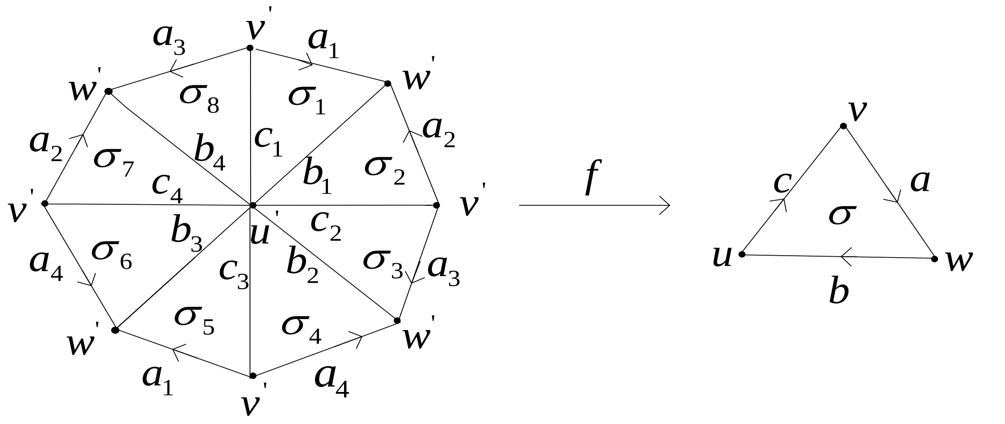

Figure 2 Extension of $f: N_3 \to P_3$ round vertex u'

Conversely, the identification of edges of the extension figure round u'in Fig(2) gives a construction of the surface N_3 and a regular folding which sends u' to u, v' to v, w' to w, a_i to a, b_i to b and c_i to c.

Note that from Fig.(2), we can construct also the extensions figures round vertex v' and vertex w' as in Fig.(3) and Fig.(4) without any contradictions.

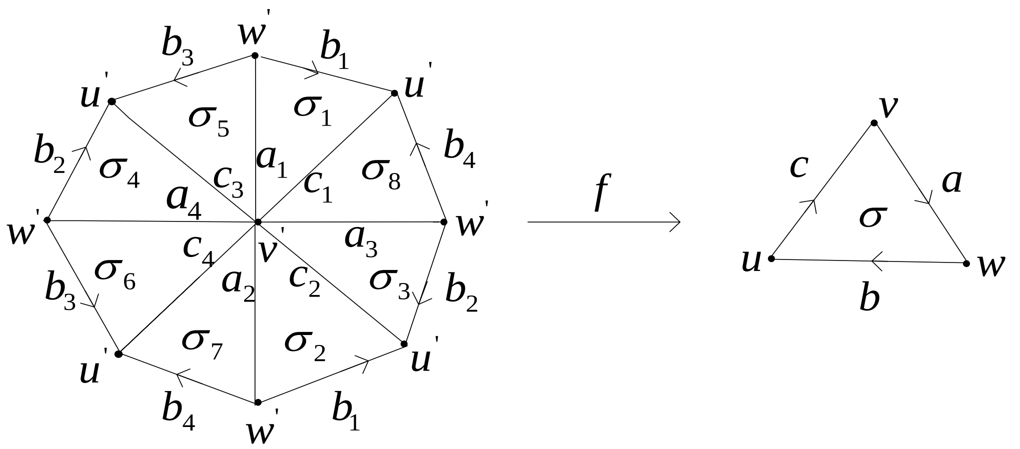

Figure 3 Extension of $f: N_3 \to P_3$ round vertex v'

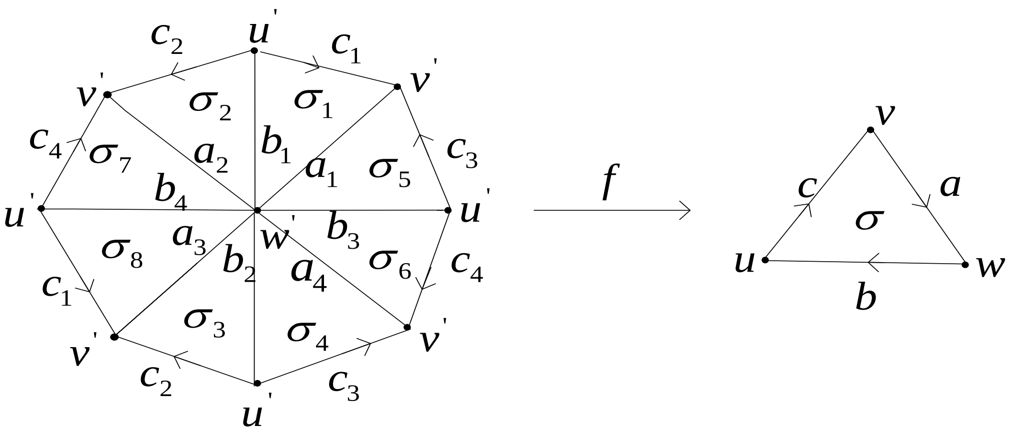

Figure 4 Extension of $f: N_3 \to P_3$ round vertex w'

One of the two types is the regular folding of $f: N_3 \to P_3$ shown in Fig.(2), which sends vertices, edges and 2-cells(triangles) of the cell decomposition of N_3as follows: $f(u') = u$, $f(a_i) = a$,

$f(v') = v$, $f(b_i) = b$, $f(\sigma_i) = \sigma$, i=1,…, 8

$f(w') = w$, $f(c_{i)} = c$, i=1,…,4

The second one is the regular folding $g: N_3 \to P_3$ shown in Fig.(5) which sends vertices, edges and 2-cells(triangles) of the decomposition of N_3 as follows:

$g(u'') = u$, $g(a_i') = a$,

$g(v'') = v$, $g(b_i') = b$, $g(\sigma_i) = \sigma$, i=1,…, 8

$g(w'') = w$, $g(c_i') = c$, i=1,…,4

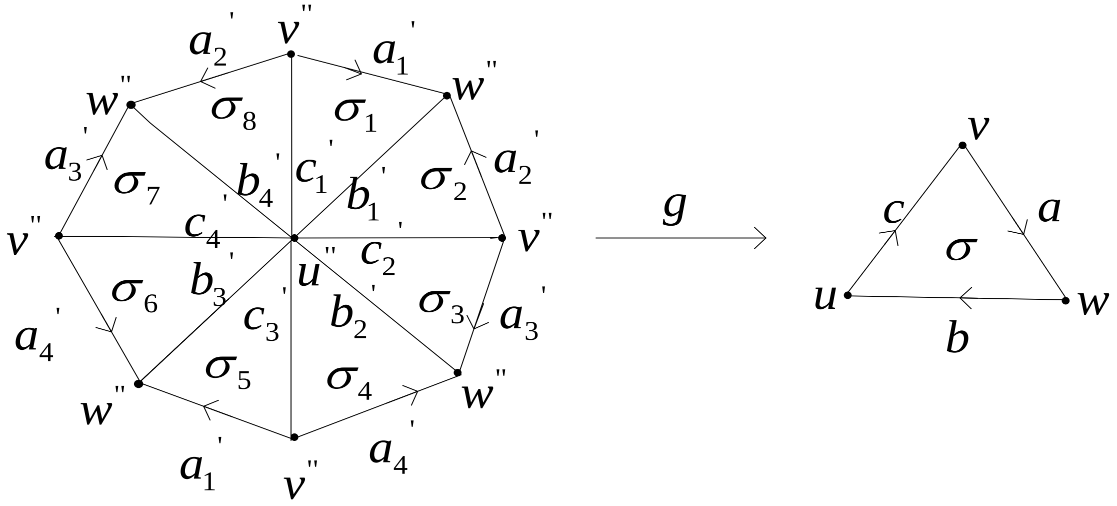

Figure 5 Extension of $g: N_3 \to P_3$ round vertex u''

Figures 6 and 7 are extensions figures of g round vertex v'' and w'' respectively.

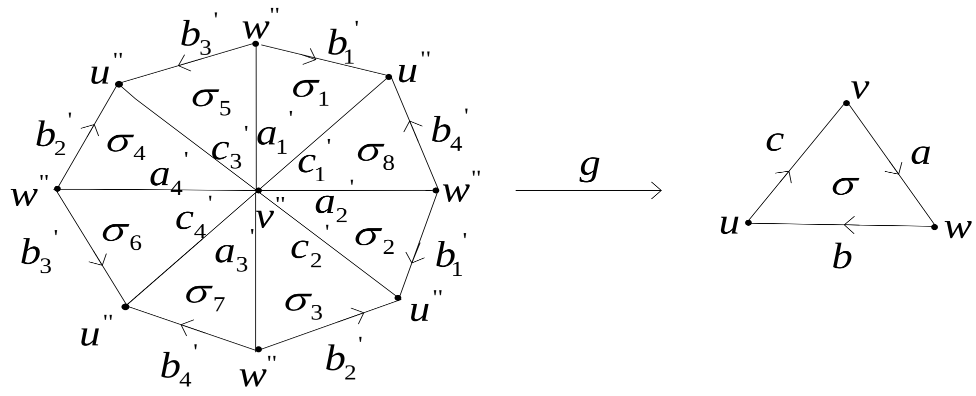

Figure 6 Extension of $g: N_3 \to P_3$ round vertex v''

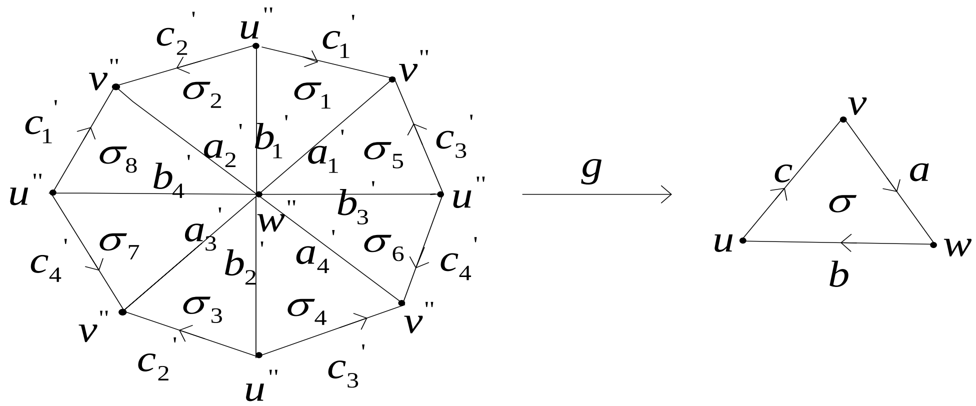

Figure 7 Extension of $g: N_3 \to P_3$ round vertex w''

These two regular foldings f and g are not topologically equivalent as we see as follows. Suppose f and g are topologically equivalent, then there is a homeomor- phism $h: N_3 \to N_3$ such that $h(\Gamma_f) = \Gamma_g$. So the vertex u' must be send by h to one of the vertices u'' and v''or w''.

If $h(u') = u''$, then the identification of edges in Fig.(2) and Fig.(6) must be the same, but which is not the case. The other two cases where $h(u') = v''$ or $h(u') = w''$, also cannot happen with the same reason. This prove that there is no such homeomorphism and therefore f and g are not topologically equivalent.

Now, consider the regular folding $g: N_3 \to P_3$. We see from the extension figures (5), (6) and (7) that there is no isometric homeomorphism $h \epsilon G(g)$ which sends $\sigma_1$to σ_2. This fact can be seen as follows. Let $h \epsilon G(g)$, then h must leave vertex u'' fixed. For if h carries vertex $u''$to vertex v'', then the identification of edges of the extension figure round u'', must be the same as that of the figure round vertex v'' , which is not the case. With the same reason, h does not carries vertex u'' to vertex w'' .

So if there is an isometry $h \epsilon G(g)$ such that $h(\sigma_1) = \sigma_2$, then h must be either the rotation round vertex u'' which sends $\sigma_1$to σ_2 or the reflection in the line containing the common edge of $\sigma_1$and σ_2, both of them damage the identification of boundary edges of the extension figure round vertex u''.Hence there is no such element of $G(g)$ that carries $\sigma_1$to σ_2. Thus $G(g)$does not acts transitively on the set of the 2-cells of C_g.

It follows from this example that every perfect folding is regular and the converse is not true.

4. NON EXISTENCE THEOREMS FOR PERFECT FOLDING

Using the properties of Theorem (2-2) and Corollary(2-3) , we obtain many non existence theorems for perfect foldings.

Theorem(4-1)

There is no perfect folding $f: M_g \to P_n$, if n and g satisfy one of the following properties:

(i) $n > 2g + 2$, $g \geq 1$

(ii) $n = 2g - m$, $g \geq m + 4$, $m \geq -1$

(iii) $n = g - m$, $g \geq 3m + 11$, $m \geq -1$

(iv) $n = g + m$, $g \geq m - 1$, $m \geq 4$

Proof:

(i) To prove (i) , and since each perfect folding is a regular folding, see Theorem (1.6) in [2].

(ii) The case $n = 2g - m$, $g \geq m + 4$, $m \geq -1$

(a) If $m = -1$, then $n = 2g + 1$, $g \geq 3$, then from (i) in Corollary(2-3), we have $\frac{\alpha}{n} = \frac{4(g-1)}{(k-2)(n-2)-4}$.

If k=4, then $\frac{\alpha}{n} = \frac{4(g-1)}{2(2g-1)-4} = \frac{2g-2}{2g-3} = \frac{q}{q-1}$, where $q = 2g - 2$, $g \geq 3$.

So $\frac{\alpha}{n}$ is not an integer for all $q \geq 4$ $(g \geq 3)$, which contradicts (iv) in Theorem(2-2). For all $k \geq 6$, we have $(k-2)(n-2) - 4 = (k-2)(2g-1) - 4 \geq 4(2g-1) - 4 = 8(g-1) > 4(g-1)$,

which contradicts (ii) in Corollary(2-3). So there is no perfect folding $f: M_g \to P_n$ if $n = 2g + 1$, $g \geq 3$.

(b) If $m = 0$, then $n = 2g$, $g \geq 4$. Now $\frac{\alpha}{n} = \frac{2(g-1)}{(k-2)(g-1)-2}$.

If k=4, then $\frac{\alpha}{n} = \frac{4(g-1)}{2(2g-2)-4} = \frac{g-1}{g-2} = \frac{q}{q-1}$, where $q = g - 1$, $g \geq 4$.

So $\frac{\alpha}{n}$ is not an integer for all $q \geq 3$ $(g \geq 4)$, which contradicts (iv) in Theorem(2-2). For all $k \geq 6$, we have $(k-2)(n-2) - 4 = (k-2)(2g-2) - 4 \geq 4(2g-2) - 4 = 4(g-1) + 2(g-2) + 2g \geq 4(g-1) + 2(g-2) + 8 > 4(g-1) + 2(g-2) \geq 4(g-1) + 4 > 4(g-1)$,

which contradicts (ii) in Corollary(2-3). This proves that there is no perfect folding $f: M_g \to P_n$ if $n = 2g$, $g \geq 4$.

(c) If $m = 1$, then $n = 2g - 1$, $g \geq 5$. Now, if k=4 , then $\frac{\alpha}{n} = \frac{4(g-1)}{(k-2)(n-2)-4} = \frac{4(g-1)}{2(2g-3)-4} = \frac{2g-2}{2g-5} = \frac{q}{q-3}$, where $q = 2g - 2$, $g \geq 5$.

So $\frac{\alpha}{n}$ is not an integer for all $q \geq 8$ $(g \geq 5)$, which contradicts (iv) in Theorem(2-2).

For all $k \geq 6$, we have $(k-2)(n-2) - 4 = (k-2)(2g-3) - 4 \geq 4(2g-3) - 4 = 4(g-1) + 4(g-3) \geq 4(g-1) + 8 > 4(g-1)$,

which contradicts (ii) in Corollary(2-3). This proves that there is no perfect folding $f: M_g \to P_n$ if $n = 2g - 1$, $g \geq 5$.

The same argument can be used to prove that there is no perfect folding $f: M_g \to P_n$ if $n = 2g-2$, $g \geq 6$ or $n = 2g-3$, $g \geq 7$ or $n = 2g-4$, $g \geq 8$ .In general there is no perfect folding $f: M_g \to P_n$ if $n = 2g-m$ $g \geq m+4$, $m \geq -1$.

(iii) The case $n = g-m$, $g \geq 3m+11$, $m \geq -1$

(a) If $m = -1$, then $n = g+1$, $g \geq 8$, then from (i) in Corollary (2-3), we have $\frac{\alpha}{n} = \frac{4(g-1)}{(k-2)(n-2)-4}$.

If k=4, then $\frac{\alpha}{n} = \frac{4(g-1)}{2(g-1)-4} = 2\,\frac{g-1}{g-3} = \frac{2q}{q-2}$, where $q = g-1$, $g \geq 8$.So $\frac{\alpha}{n}$ is not an integer for all $q \geq 7 (g \geq 8)$, which contradicts(iv) in Theorem(2-2).

Also, if k=6, then $\frac{\alpha}{n} = \frac{g-1}{g-2}$, which is not an integer for all $g \geq 8$, and hence contradicts (iv) in Theorem(2-2).

For all $k \geq 8$, we have $(k-2)(n-2)-4 = (k-2)(g-1)-4 \geq 6(g-1)-4 = 4(g-1)+(2g-6) \geq 4(g-1)+10 > 4(g-1)$,

which contradicts (ii) in Corollary(2-3). So $\frac{\alpha}{n}$ is not an integer and hence there is no perfect folding $f: M_g \to P_n$ if $n = g+1$, $g \geq 8$.

(b) If $m = 0$, then $n = g$, $g \geq 11$ .Now , $\frac{\alpha}{n} = \frac{4(g-1)}{(k-2)(g-2)-4}$.

If k=4, then $\frac{\alpha}{n} = \frac{4(g-1)}{2(g-2)-4} = \frac{2g-2}{g-4}$, which is not an integer for all $g \geq 11$, which contradicts(iv) in Theorem(2-2).

If k=6, then $\frac{\alpha}{n} = \frac{g-1}{g-3}$, which is not an integer for all $g \geq 11$.

For all $k \geq 8$ we have $(k-2)(n-2)-4 = (k-2)(g-2)-4 \geq 6(g-2)-4 = 4(g-1)+2(g-6) \geq 4(g-1)+10 > 4(g-1)$,

which contradicts (ii) in Corollary(2-3). This proves that there is no perfect folding $f: M_g \to P_n$ if $n = g$, $g \geq 11$.

(c) If $m = 1$, then $n = g-1$, $g \geq 14$ .If k=4, we have $\frac{\alpha}{n} = \frac{4(g-1)}{2(g-3)-4} = \frac{2g-2}{2g-5}$, which is not an integer for all $g \geq 14$, which contradicts (iv) in Theorem(2-2).

If k=6, then $\frac{\alpha}{n} = \frac{g-1}{g-4}$, which is not an integer for all $g \geq 14$ and hence contradicts (iv) in Theorem(2-2).

For all $k \geq 8$, we have $(k-2)(n-2)-4 = (k-2)(g-3)-4 \geq 6(g-3)-4 = 4(g-1)+2(g-9) \geq 4(g-1)+10 > 4(g-1)$,

which contradicts (ii) in Corollary(2-3). Hence there is no perfect folding $f: M_g \to P_n$ if $n = g-1$, $g \geq 14$.

By using the same argument we can prove that there is no perfect folding $f: M_g \to P_n$ if $n = g-2$, $g \geq 17$ or $n = g-3$, $g \geq 20$ or $n = g-4$, $g \geq 23$ .In general there is no perfect folding $f: M_g \to P_n$ if $n = g-m$ $g \geq 3m+11$, $m \geq -1$.

(iv) The case $n = g+m$, $g \geq m-1$, $m \geq 4$

(a) If $m = 4$, then $n = g+4$, $g \geq 3$. Now, $\frac{\alpha}{n} = \frac{4(g-1)}{(k-2)(g+2)-4}$.

If k=4, then $\frac{\alpha}{n} = 2\,\frac{g-1}{g}$, which is not an integer for all $g \geq 3$, which contradicts(iv) in Theorem(2-2).

For all $k \geq 6$, we have $(\mathrm{k}-2)(\mathrm{n}-2)-4=(\mathrm{k}-2)(\mathrm{g}+2)-4 \geq 4(\mathrm{g}+2)-4=4(\mathrm{g}-1)+8>4(g-1)$,

which contradicts (ii) in Corollary(2-3). Hence there is no perfect folding $f: M_g \rightarrow P_n$ if $n=g+4$, $g \geq 3$.

(b) If $m=5$, then $n=g+5$, $g \geq 4$.

If k = 4, then $\frac{\alpha}{n}=\frac{4(g-1)}{2(g+3)-4}=\frac{2(g-1)}{g+1}$, which is not an integer for all $g \geq 4$, which contradicts (iv) in Theorem(2-2).

For all $k \geq 6$, we have $(\mathrm{k}-2)(\mathrm{n}-2)-4=(\mathrm{k}-2)(\mathrm{g}+3)-4 \geq 4(\mathrm{g}+3)-4=4(\mathrm{g}-1)+12>4(g-1)$,

which contradicts (ii) in Corollary(2-3). Hence there is no perfect folding $f: M_g \rightarrow P_n$ if $n=g+5$, $g \geq 4$.

In this way we can prove that there is no perfect folding $f: M_g \rightarrow P_n$ if $n=g+m$ $g \geq m-1$, $m \geq 4$. □

Theorem(4-2)

There is no perfect folding $f: N_p \rightarrow P_n$ if n and p satisfy one of the following properties:

(i) $n>p+2$, $p \geq 2$

(ii) $n=p-m$, $p \geq 2m+7$, $m \geq -1$

Proof:

(i) To prove (i), and since each perfect folding is a regular folding, see Theorem (1.6) in [2].

(ii) The case $n=p-m$, $p \geq 2m+7$, $m \geq -1$

(a) If $m=-1$, then $n=p+1$, $p \geq 5$. Now from (iv) in Corollary(2-3), we have $\frac{\alpha}{n}=\frac{2(p-2)}{(k-2)(n-2)-4}$.

If k=4, then $\frac{\alpha}{n}=\frac{2(p-2)}{2(p-1)-4}=\frac{p-2}{p-3}=\frac{q}{q-1}$,where $q=p-2$, $p \geq 5$,

which is not an integer for all $q \geq 3$ $(p \geq 5)$, which contradicts (iv) in Theorem(2-2).

For all $k \geq 6$, we have $(k-2)(n-2)-4=(k-2)(p-1)-4 \geq 4(p-1)-4=4(p-2)>2(p-2)$,

which contradicts (v) in Corollary (2-3), and hence there is no perfect folding $f: N_p \rightarrow P_n$ if $n=p+1$, $p \geq 5$.

(b) If $m=0$, then $n=p$, $p \geq 7$, and we have $\frac{\alpha}{n}=\frac{2(p-2)}{(k-2)(p-2)-4}$.

If k=4, then $\frac{\alpha}{n}=\frac{p-2}{p-4}=\frac{q}{q-2}$, where $q=p-2$, $p \geq 7$, which is not an integer for all $q \geq 5$ $(p \geq 7)$, which contradicts (iv) in Theorem(2-2).

For all $k \geq 6$, we have $(k-2)(n-2)-4=(k-2)(p-2)-4 \geq 4(p-2)-4=2(p-2)+2(p-4) \geq 2(p-2)+6>2(p-2)$,

which contradicts (v) in Corollary(2-3), and hence there is no perfect folding $f: N_p \rightarrow P_n$ if $n=p$, $p \geq 7$.

(c) If $m=1$, then $n=p-1$, $p \geq 9$, and we have $\frac{\alpha}{n}=\frac{2(p-2)}{(k-2)(p-3)-4}$.

If k=4, then $\frac{\alpha}{n} = \frac{2(p-2)}{2(p-3)-4} = \frac{p-2}{p-5} = \frac{q}{q-3}$, where $q = p - 2$, $p \geq 9$, so $\frac{\alpha}{n}$ is not an integer for all $q \geq 7$ $(p \geq 9)$, which contradicts (iv) in Theorem(2-2).

For all $k \geq 6$, we have $(k-2)(n-2) - 4 = (k-2)(p-3) - 4 \geq 4(p-3) - 4 = 2(p-2) + 2(p-6) \geq 2(p-2) + 6 > 2(p-2)$,

which contradicts (v) in Corollary(2-3). Hence there is no perfect folding $f: N_p \to P_n$ if $n = p - 1$, $p \geq 9$.

The same argument can be used to prove that there is no perfect folding $f: N_p \to P_n$ if $n = p - 2$, $p \geq 11$ or $n = p - 3$, $p \geq 13$ .In general there is no perfect folding $f: N_p \to P_n$ if $n = p - m$, $p \geq 2m + 7$, $m \geq -1$. □

Theorem (4-3)

There is no perfect folding $f: N_p \to P_n$ if n is even and p is an odd number

Proof:

From Proposition (1-1) and Theorem (2-2), we have

$p = 2 + \frac{(k-2)(n-2)-4}{2n}\alpha = 2 + \frac{1}{2}[(k-2)(n-2) - 4]l$, $l = \frac{\alpha}{n}$. Since l is a positive integer and n is even, the right hand side of the equality must be even, which contradicts the assumption that p is an odd number. □

Theorem (4-4)

There is no perfect folding $f: M_g \to P_n$ if n is even and g is even number

Proof:

From Proposition (1-1) and Theorem (2-2), we have

$g = 1 + \frac{(k-2)(n-2)-4}{4n}\alpha = 1 + \frac{1}{4}[(k-2)(n-2) - 4]l$, $l = \frac{\alpha}{n}$. Since l is a positive integer and n is even, the right hand side of the equality must be odd, which contradicts the assumption that g is even number. □

5. CONCLUSION

Let $f: M \to P_n$ be a regular folding with α vertices, β edges and γ faces. Consider the quintuplet $(k, n, \alpha, \beta, \gamma)$ associated to a regular folding of M onto P_n , where k is the valency of regularity. In this paper we discover a new additional relation that must be satisfied by the quintuplet $(k, n, \alpha, \beta, \gamma)$. Using this we obtain non existence theorems for perfect (regular) folding between a wide range of pairs of surfaces and polygons.

CONJECTURE

For all quintuplet $(k, n, \alpha, \beta, \gamma)$ satisfying the conditions in Proposition (1-1) and Theorem (2-2), there exist always regular folding with quintuplet $(k, n, \alpha, \beta, \gamma)$.

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